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External Noise and the Origin and Dynamics of Structure in Convectively Unstable Systems

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Some basic concepts and earlier work on external noise and the convectively unstable Ginzburg–Landau equation are reviewed, and some of the ideas presented in the earlier work are investigated further and extended. In particular, further consideration is given to convective chaos—chaos which only occurs in a moving frame of reference; and slugs—localized structures which are surrounded by a stable stationary state. Some new results on secondary convective instabilities and on periodic systems with a spatially varying instability are discussed. Work on the coupled Ginzburg–Landau equation is reviewed. Actual physical systems are discussed.

KEY WORDS: Convective instability; noise-sustained structure; spatiotemporal intermittency; convective chaos; slugs; additive noise; Ginzburg–Landau equation; pattern selection; open-flow systems; secondary instability.

1. INTRODUCTION

Consider the stationary state of some spatially extended system and a small spatially localized perturbation about this state. If the perturbation grows with time at a given stationary point, the stationary state is *absolutely* unstable. However, if the perturbation travels such that the perturbation grows only in a moving frame of reference, eventually damping at any given stationary point, the stationary state is *convectively* unstable.[1–8] Although this distinction is not often explicitly made, it is an important distinction since, in convectively unstable systems: (1) continuous external noise (or other continuous external perturbation) is necessary for an asymptotic state different from the stationary state—giving rise to a *noise-sustained structure*[1,2]; (2) the external noise is selectively and spatially amplified, forming spatially growing waves—this being a mechanism for

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pattern selection[1,2]; and (3) the external noise can play an important role in macroscopic dynamics such as *spatiotemporal intermittency*.[1,2,9] Since any system with nonzero group velocity will be convectively unstable sufficiently close to and above onset of the instability,[10] the behavior reviewed and studied in this paper is expected to be very common in nature.

Since the equations describing a physical system may be very complicated and difficult to study, approximations are sometimes made to reduce the equations to a simpler set of equations. These equations will be more tractable for study while, it is hoped, preserving the essential features under study. One equation to which many physical systems can be reduced is the time-dependent generalized Ginzburg–Landau equation.[11–15] This equation results from an expansion in some parameter (e.g., the Reynolds, Rayleigh, or Taylor number for a fluid system) near the critical value of that parameter. For example, Rayleigh–Bénard convection (fluid with a vertical temperature gradient),[11] plane Poiseuille flow (fluid flowing between two parallel plates),[13] and wind-induced water waves[14] can be reduced to the Ginzburg–Landau equation. Since the Ginzburg–Landau equation is a generic partial differential equation which exhibits much of the spatiotemporal phenomenon seen in actual physical systems, it is an ideal equation for study and will be used as a model equation for most of the studies here. Also, even though the Ginzburg–Landau equation is one dimensional, studies of this equation can provide insight and intuition about higher-dimensional systems and can suggest possible numerical or experimental studies of actual physical systems.

In this paper I will review some basic concepts and some earlier work on the Ginzburg–Landau equation, as well as look in more detail at and extend some of the ideas presented in the earlier work. In particular, I will look in more detail at *convective chaos*—chaos which only occurs in a moving frame of reference (see Section 4); and *slugs*—localized structures which are surrounded by a stable stationary state (see Section 5). I will also look at some new results on secondary convective instabilities and on periodic systems with a spatially varying instability. I will then review work on the coupled Ginzburg–Landau equation.[10] Throughout and toward the end of this paper I will discuss actual physical systems.

2. THE GINZBURG–LANDAU EQUATION

The Ginzburg–Landau equation is

$$\frac{\partial \psi}{\partial t} = a\psi - v_g \frac{\partial \psi}{\partial x} + b \frac{\partial^2 \psi}{\partial x^2} - c\, |\psi|^2\, \psi \tag{1}$$

where the dependent variable $\psi(x, t)$ is in general complex, the coefficients a, b, and c are in general complex, $b_r > 0$, and v_g is the group velocity (of the most unstable mode). Real and imaginary parts will be denoted by the subscripts r and i, respectively, or by Re and Im, respectively. The variable ψ is related to the actual system by being the slowly varying amplitude of a plane wave about the stationary state of the system. If a_r is positive, a small perturbation will grow with time. In addition, if $|v_g|$ is sufficiently large (see Section 3), the perturbation will be convected out of any given region and the system will be convectively unstable. If c_r is positive, the perturbation will eventually saturate and produce some pattern.

To give the reader a better indication of the relation of ψ to the actual variable in a physical system, I note that for two-dimensional plane Poiseuille flow (flow between two parallel planes) the stream function ξ to first order in $\varepsilon = (R - R_c)^{1/2}$ (where R is the Reynolds number and R_c is the critical Reynolds number) is[13]

$$\xi(x, y, t) = y - y^3/3 + 2\varepsilon \, \mathrm{Re}[\psi(\chi, \tau) \, \phi(y) \, e^{i(k_c x - \omega_c t)}] \tag{2}$$

Here x and y are the spatial coordinates parallel and perpendicular to the planes, respectively; $\chi = \varepsilon x$ and $\tau = \varepsilon^2 t$ are slowly varying space and time variables; k_c and ω_c are the wavenumber and frequency corresponding to the critical Reynolds number R_c; $y - y^3/3$ is the unperturbed laminar state which corresponds to a parabolic velocity profile; and $\phi(y)$ is the eigenfunction of the most unstable mode for the Orr–Sommerfeld equation, which is the linear stability equation for fluid flow. In Eq. (1) (with x and t replaced by χ and τ, respectively) a and c are of order ε^2, and v_g and b are of order ε^0. The velocity components parallel and perpendicular to the planes, respectively, are $u = -\partial\xi/\partial y$ and $v = \partial\xi/\partial x$. Since $\psi(\chi, \tau)$ is the slowly varying amplitude of a plane wave, Eq. (1) is often referred to as an amplitude equation.

3. CONVECTIVE INSTABILITY AND NOISE-SUSTAINED STRUCTURE

Consider a small, spatially localized perturbation about the stationary state of some spatially extended system. Figure 1 shows three distinct types of behavior that the small localized perturbation can undergo. If the state is absolutely stable, the perturbation will be damped in any frame of reference. If the state is absolutely unstable, the edges of the perturbation will move in opposite directions and therefore the perturbation will grow at any given stationary point. If the state is convectively unstable, the edges of the perturbation will move in the same direction and therefore the pertur-

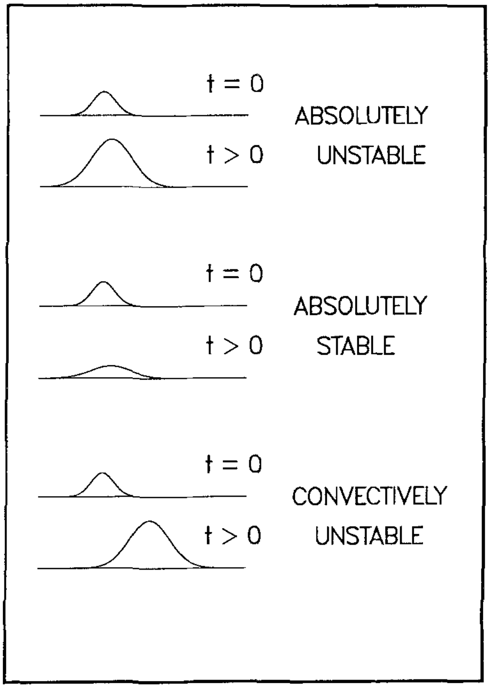

Fig. 1. Illustration of three types of behavior for a perturbation. The convectively unstable case results in the amplification of noise. Reprinted from ref. 2.

bation will grow only in a moving frame of reference, eventually damping at any given stationary point.

In the absolutely unstable case, the perturbation will grow and saturate (because of nonlinearities), producing in the asymptotic time limit some pattern (which may be changing with time). Therefore, a single perturbation produces a pattern for all time. In contrast, in the convectively unstable case, the perturbation and resulting pattern will move spatially such that the pattern eventually leaves the boundaries of the system. Therefore, a single perturbation produces only a temporary pattern and in the asymptotic time limit the system returns to the stationary state. In order to have a permanent pattern in a convectively unstable system, it is therefore necessary to continuously perturb the system. If the system is continuously perturbed by external noise, we refer to the resulting pattern as a *noise-sustained structure* and the corresponding state as a *noise-sustained state.*[1,2] In general the noise will be *selectively* amplified by the dynamics (i.e., some wavenumbers will be more amplified than others). This will give rise to spatially growing waves and will be responsible for the selection of the wavelength of any regular pattern that may form. Since the noise will be amplified exponentially, the natural noise inherent in a physical system should in many cases be sufficient to produce a noise-sustained structure.

To make these ideas more concrete, let us now consider the Ginzburg–Landau equation (1). The state $\psi = 0$ of Eq. (1) may be shown[1,2] to be *absolutely stable* if

$$a_r < 0 \tag{3}$$

absolutely unstable if

$$a_r - \frac{v_g^2 b_r}{4\,|b|^2} > 0 \tag{4}$$

and *convectively unstable* if

$$a_r - \frac{v_g^2 b_r}{4\,|b|^2} < 0 \qquad \text{and} \qquad a_r > 0 \tag{5}$$

Figure 2 shows $\psi_r(x, t)$ plotted as a function of x at five different times for a set of parameter values in the convectively unstable regime. A low level of noise is introduced at the left boundary by letting $\psi_r(0, t)$ and $\psi_i(0, t)$ equal random numbers uniformly distributed between $-r$ and r. The initial conditions were $\psi(x, 0) = 0$. The boundary condition at the right boundary is $d^2\psi/dx^2 = 0$ to simulate an open boundary. For the numerical method see refs. 1 and 2. Figure 2a is at an early time and shows the structure in the process of formation. Figures 2b–2d show the structure at large times after the system has reached a statistically steady state. External noise at the left boundary is selectively and spatially amplified, giving rise to spatially growing waves. When the amplitude of the wave becomes sufficiently large, the waves saturate, producing the structure (i.e., pattern). The selective and spatial amplification of the noise is the pattern selection mechanism responsible for the sinusoidal portion of the structure. Because the sinusoidal structure is itself convectively unstable (i.e., a secondary convective instability), it breaks up at some spatial point, producing an irregular structure. Since there are irregularities in the spatially growing waves that change with time, the point at which the structure breaks up changes with time (compare Figs. 2b, 2c, and 2d), resulting in random *spatiotemporal intermittency*.

At $t = 116$ the noise is removed. Figure 2e shows the structure at $t = 300$. We see that the structure moves out through the right boundary. As t increases further, the structure will continue moving to the right until the state returns to $\psi = 0$ everywhere. Therefore, the structure seen in Figs. 2b–2d is indeed a noise-sustained structure.

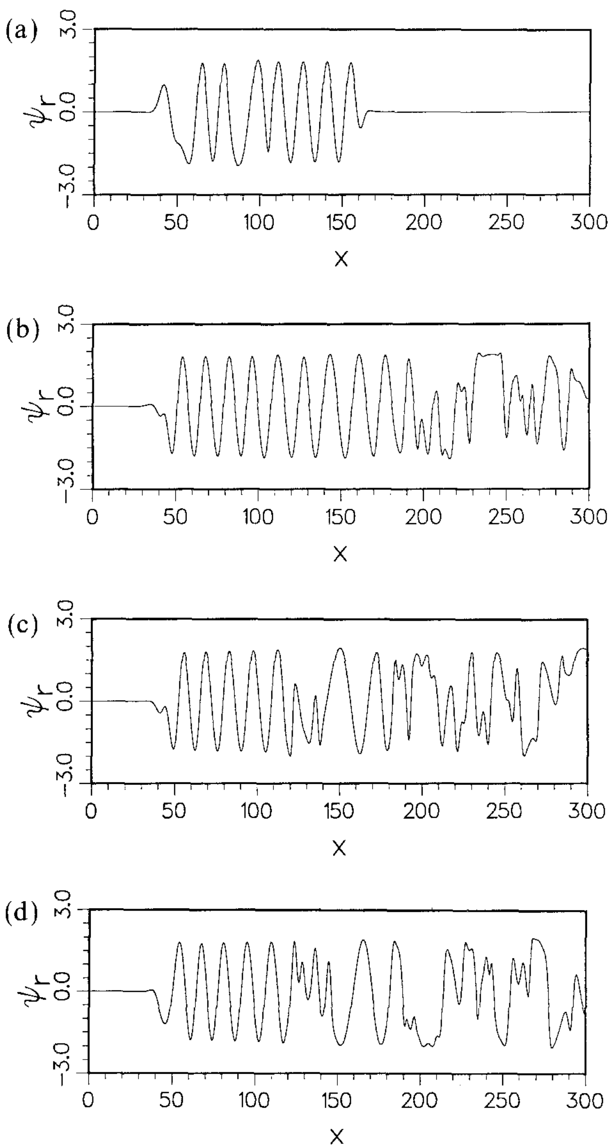

Fig. 2. Plot of ψ_r as a function of x at (a) $t=20$, (b) $t=100$, (c) $t=115$, (d) $t=116$, and (e) $t=300$. The parameter values in Eq. (1) were $a=2$, $v_g=5.2$, $b_r=1.8$, $b_i=-1$, $c_r=0.5$, and $c_i=1$. The initial condition was $\psi(x,0)=0$ and the noise level at the left boundary was $r=10^{-7}$. At $t=116$ the noise was removed. The external noise at the left boundary is selectively and spatially amplified, resulting in the observed pattern (panels b–d). Spatiotemporal intermittency is observed. Without the noise (panel e) the pattern moves out through the right boundary.

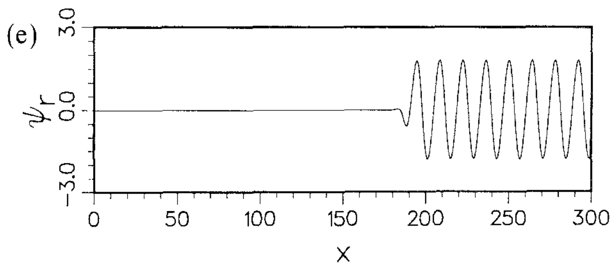

Fig. 2 (*continued*)

Since eigenvalues are often used to calculate whether or not the stationary state of a system is unstable, a few words should be mentioned about this method for the calculation of stability with regard to convective instability. If one calculates the eigenvalues of the linearized equation with the given set of boundary conditions for a convectively unstable state, one will find that the most unstable eigenvalue (i.e., λ_g) has a real part that is negative. This is so since a perturbation will be convected out of the system and the system will return to the stationary state. For example, the most unstable eigenvalue of the linearized equation (1) with boundary conditions $\psi = 0$ at $x = 0$ and $x = L$, where L is the length of the system, is $\lambda_g = a - v^2/(4b) - \pi^2 b/L^2$. For $L^2 \gg \pi^2 b_r$, $\text{Re}[\lambda_g] < 0$ is then equivalent to the first part of Eq. (5). Therefore, if the real part of the eigenvalue is negative, the state may be either absolutely stable or convectively unstable, and another test will be needed to distinguish between these two alternatives. One method is to then calculate the most unstable eigenvalue with periodic boundary conditions imposed instead (i.e., λ_p).[1,2] For example, the most unstable eigenvalue of the linearized equation (1) with periodic boundary conditions is $\lambda_p = a$, and $\text{Re}[\lambda_p] > 0$ is equivalent to the second part of Eq. (5). Therefore, if the real part of the most unstable eigenvalue with the given set of boundary conditions is negative (i.e., $\lambda_g < 0$), and the real part of the most unstable eigenvalue with periodic boundary conditions imposed instead is positive (i.e., $\lambda_p > 0$), the state will be convectively unstable. This follows from the fact that a perturbation which would have otherwise traveled out of the system with the given set of boundary conditions will be fed back into the system if periodic boundary conditions are imposed instead. If $\lambda_g > 0$, the state will be absolutely unstable. If $\lambda_p < 0$, the state will be absolutely stable.

Even though the equations describing a system may be Galilean invariant, the equations *plus* boundaries (assuming the boundaries are not periodic) will not be. Therefore, the above effect is not something that can be transformed away. Also, because of boundaries there *is* a preferred frame of reference. For example, in fluid flow over a flat plate[16] the preferred frame of reference is that in which the plate is at rest. Perturbations near the leading edge of the plate will be amplified as they are convected over the surface of the plate. It may also happen that the preferred frame of reference is not the laboratory frame of reference. For example, if a plate were being towed through some fluid, the preferred frame of reference would be that in which the plate is at rest. Another example is side branching in a dendrite (see Section 9). The preferred frame of reference is that in which the tip of the dendrite is at rest, since it is the perturbations at the tip that are being amplified as they are convected along the sides of the dendrite (relative to the tip).

Also, the above effect is very different from that occurring in an initial value problem, where some initial noise in a system is amplified, giving rise to some pattern. To reiterate, in a convectively unstable system, continuous external noise is selectively and spatially amplified as it is convected along, giving rise to spatially growing waves. Therefore, the formation of the spatially growing waves is dependent on the constant flux of new information and is not something that can be described by an initial value problem. That is not to say that there is no behavior in a convectively unstable system that can be described by an initial value problem. For example, if the system is given an initial spatially localized perturbation, one can consider a finite region which contains the perturbation and allow this region to move with the perturbation. In this case, the behavior will be described quite well by an initial value problem as long as the perturbation stays confined to the region under consideration and until the region hits the boundary of the system.

An interesting point to consider is that of prediction in convectively unstable systems. The situation is much worse than the problems discussed by Lorenz concerning deterministic chaos.[17] For, in a convectively unstable system prediction for even short times is not possible, unless the microscopic noise is known. For example, prediction of the spatiotemporal intermittency seen in Figs. 2c and 2d is not possible unless the microscopic external noise at the left boundary is known. Another example is fluid flow over a flat plate.[16] Small-scale fluctuations near the leading edge of the plate are amplified as they are convected over the surface of the plate. Further downstream turbulent spots form randomly in space and time. Unless the fluctuations near the leading edge are known, prediction of the occurrence of the turbulent spots is not possible.

As noted in the introduction, any system with nonzero group velocity will be convectively unstable sufficiently close to and above onset of the instability.[10] This may be seen by noting that, slightly above onset of the instability, the velocities of both edges of the perturbation will have the

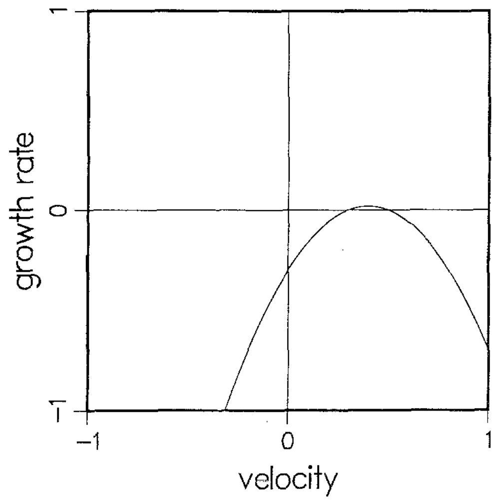

Fig. 3. Plot of the growth rate of a perturbation as a function of the frame of reference from which the perturbation is observed when the system is slightly above onset. Since the velocity of both edges of the perturbation is the same sign, the system is *convectively unstable.*

same sign as the group velocity (since the group velocity is nonzero). This is illustrated in Fig. 3, which shows the growth rate of a perturbation as a function of the velocity of the frame of reference from which the perturbation is observed when the system is slightly above onset. Since the velocities of both edges of the perturbation have the same sign, the perturbation will be convected out of any given region. The velocities of the edges are given by those velocities for which the growth rate is zero, and the group velocity is given by that velocity for which the growth rate is maximum.[8] By group velocity in the above we mean the group velocity of the most unstable mode, since this will be the velocity at which a perturbation will grow most rapidly in the asymptotic time limit and since this velocity will correspond to "moving with the perturbation."

Also, the above observation may be seen (although less generally) by referring to Eq. (5). If v_g is nonzero and if a_r is only slightly positive, Eq. (5) will be satisfied and the system will be convectively unstable. At this point it may also be worth mentioning that by slightly above onset I mean slightly above onset for the actual physical system and not for the corresponding Ginzburg–Landau equation. For, because of boundary effects, the point of onset for the actual physical system may be slightly different from the point of onset for the corresponding Ginzburg–Landau equation.

As the control parameter is changed further from its critical value, it may also happen in some systems that the system will make a transition from convectively unstable to absolutely unstable. In other systems, such as plane Poiseuille flow, the system is convectively unstable for all values of the control parameter (which is the Reynolds number for plane Poiseuille flow) for which the system is unstable.[8]

4. CONVECTIVE CHAOS

Chaos is characterized by the exponential average separation of nearby states. However, since perturbations can grow or decay depending on the frame of reference, the divergence or convergence of nearby states can also depend on the frame of reference, meaning that the usual concept of chaos needs to be modified when considering spatially extended systems with nonzero group velocity. Just as we perturbed about the stationary state of a system to determine whether the state were absolutely or convectively unstable, we can also perturb about the general irregular (e.g., turbulent) state of a system to determine whether the state is absolutely or convectively chaotic.[2,18] Consider the general irregular state of some system and a small, spatially localized perturbation about this state. If the perturbation grows exponentially on the average at a given stationary

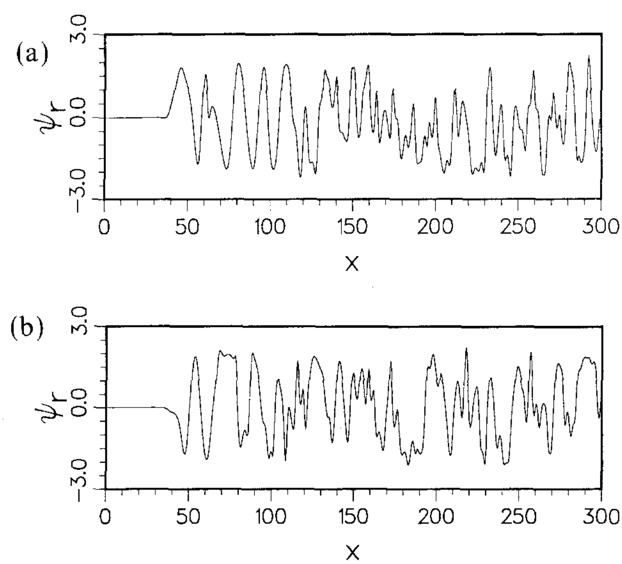

Fig. 4. Plot of ψ_r as a function of x after the system has reached a statistically steady state at two different times separated by $t = 10$. The parameter values in Eq. (1) are the same as those in Fig. 2, except for $b_r = 1$.

point, we will define the state as being *absolutely chaotic*. However, if the perturbation grows exponentially on the average only in a moving frame of reference, eventually damping at any given stationary point, we will define the state as being *convectively chaotic*. If the perturbation does not grow exponentially on the average in any frame of reference, the state is not chaotic.

The next question is how to define a measure for convective chaos. For, if we naively calculate the usual largest Lyapunov exponent for a convectively chaotic flow, we will find that the exponent is negative. This is so since perturbations about the convectively chaotic state are convected out of the system, causing nearby states to converge exponentially on the average in the asymptotic time limit. For example, in the Ginzburg–Landau equation (1), if we consider an infinitesimal perturbation $\delta\psi(x, t)$ about the convectively chaotic state $\psi(x, t)$, the perturbation will be convected out of the system, causing the states $\psi(x, t)$ and $\psi(x, t) + \delta\psi(x, t)$ to converge exponentially on the average in the asymptotic time limit, giving a negative value for the largest Lyapunov exponent. The largest Lyapunov exponent being negative for a convectively chaotic flow is similar to the real part of the most unstable eigenvalue being negative for a convectively unstable state.

These ideas will become clearer by again considering the Ginzburg–Landau equation and referring to Fig. 5, which shows the evolution of an

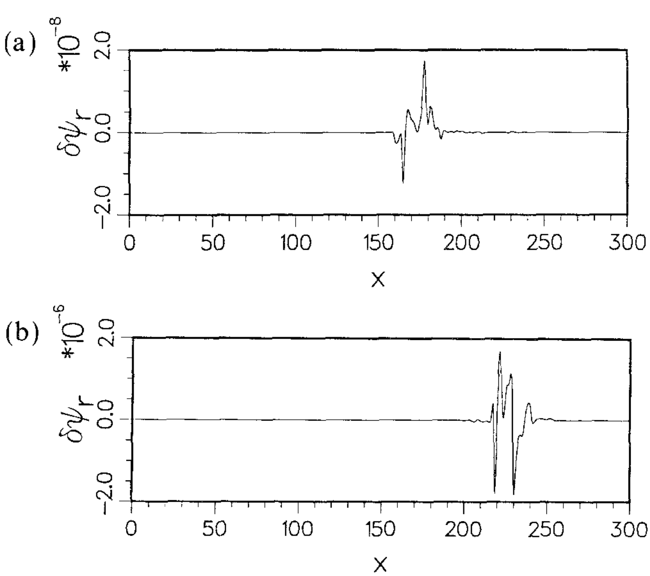

Fig. 5. Plot of an infinitesimal perturbation $\delta\psi$ about the states seen in Fig. 4. The perturbation is convected to the right as it grows, demonstrating *convective chaos*.

infinitesimal, spatially localized perturbation $\delta\psi(x, t)$ about the state $\psi(x, t)$ shown in Fig. 4, $\delta\psi$ being determined by solving the linear equation

$$\frac{\partial \delta\psi}{\partial t} = a\delta\psi - v_g \frac{\partial \delta\psi}{\partial x} + b \frac{\partial^2 \delta\psi}{\partial x^2} - 2c\, |\psi|^2\, \delta\psi - c\psi^2 \delta\psi^* \tag{6}$$

along with Eq. (1). We see that the perturbation is convected to the right as it grows, demonstrating convective chaos. For sufficiently large times, the perturbation will be convected out through the right boundary, leaving only its trailing left edge, which will decrease exponentially with time. Note that the absolute vertical scale in Fig. 5 is irrelevant, since one is solving a linear equation (i.e., one may multiply the vertical scale by any constant).

These ideas may also be demonstrated by plotting the logarithm of the separation between the two nearby states as a function of time, where the separation is given by

$$\zeta(t) = \left(\int_0^L |\delta\psi(x, t)|^2\, dx \right)^{1/2} \tag{7}$$

where L corresponds to the length of the system. Referring to Fig. 6, we see that initially the states separate corresponding to the growing perturbation seen in Fig. 5. However, for larger times the perturbation moves out

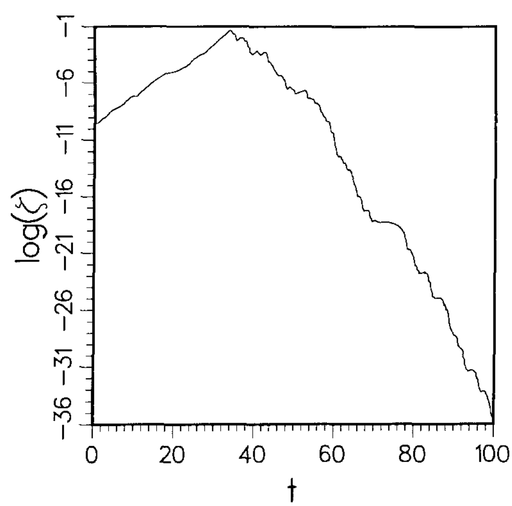

Fig. 6. Separation between two nearby states given by Eq. (7). The states initially separate corresponding to the growing perturbation seen in Fig. 5. The perturbations in Figs. 5a and 5b correspond to $t = 10$ and $t = 20$, respectively. After $t = 32$ the states converge, corresponding to the perturbation being convected out through the right boundary.

through the right boundary, leaving only its damping trailing edge, corresponding to the convergence of the states. Again note that the absolute vertical scale is irrelevant, since one is solving a linear equation (i.e., any constant may be added to the vertical scale).

In order to define a measure for convective chaos, it is necessary to take some finite spatial region Ω of the flow and allow this region to move at some velocity v. Instead of calculating the Lyapunov exponent for the entire system, we can then calculate the exponent for the moving region $\Omega(v)$ giving the Lyapunov exponent $\lambda(\Omega(v))$. Let v_m be that v which gives the largest value for $\lambda(\Omega(v))$. If $\lambda(\Omega(v_m)) > 0$, we then say that the flow is chaotic in the region $\Omega(v_m)$ and that $\lambda(\Omega(v_m))$ is a measure of that chaos.

For example, for the Ginzburg–Landau equation (1), we can take the region $\Omega(v)$ as the region extending from $x_1 + vt$ to $x_2 + vt$, i.e., the region $\{x_1 + vt, x_2 + vt\}$. Consider the initial perturbation $\delta\psi(x, 0)$ spatially localized in the region $\{x_1, x_2\}$. The subsequent evolution of this perturbation will be given by Eq. (6), where $\psi(x, t)$ is given by Eq. (1). We define the largest *velocity-dependent* Lyapunov exponent as[18]

$$\lambda(v; x_1, x_2) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\zeta(v, x_1, x_2, t)}{\zeta(v, x_1, x_2, 0)} \tag{8}$$

where

$$\zeta(v, x_1, x_2, t) = \left(\int_{x_1 + vt}^{x_2 + vt} |\delta\psi(x, t)|^2 \, dx \right)^{1/2} \tag{9}$$

Let v_m be that v which gives the maximum value for $\lambda(v; x_1, x_2)$. If $\lambda(v_m; x_1, x_2) > 0$, we say that the system is chaotic in the region $\{x_1 + v_m t, x_2 + v_m t\}$ and that $\lambda(v_m; x_1, x_2)$ is a measure of that chaos. If $v = 0$ and if x_1 and x_2 correspond to the boundaries of the system, this definition reduces to the definition of the usual largest Lyapunov exponent. For $v > 0$ the system must be extended to infinity in the positive x direction, for otherwise the region $\{x_1 + vt, x_2 + vt\}$ will hit the boundary. In practice, the system must be long enough for reasonable convergence. Also, if $x_2 - x_1$ is sufficiently large, the largest velocity-dependent Lyapunov exponent will in most cases be independent of x_1 and x_2, giving $\lambda(v)$ for the exponent.

Since experimentalists often calculate Lyapunov exponents from the reconstruction of a single time series, an important question is whether or not velocity-dependent Lyapunov exponents can be calculated from single time series. In the stationary frame of reference (i.e., $v = 0$) it is clear that a reconstruction of a single time series will not give a reasonable value for the largest exponent for a coinvectively chaotic flow. For example, the exponent from a time series taken at a stationary point in the irregular portion of the structure in Fig. 4 will be positive since the time series is aperiodic, even though $\lambda(v=0)$ is negative. This reflects the fact that there is a constant flux of new information to the right originating from the noise at the left boundary. However, in a comoving frame of reference this will no longer be a problem and the exponent from a time-series reconstruction taken at a point that is moving with the flow [e.g., moving at v_g for Eq. (1)] should give a reasonable value for the maximal largest velocity-dependent Lyapunov exponent.[18]

For many systems it would be impractical to calculate numerically the largest velocity-dependent Lyapunov exponent, since the system would need to be very long in order to get reasonable convergence. To give a criterion which may be easier to calculate for some systems we can generalize the idea of calculating the eigenvalues both for the given set of boundary conditions and for periodic boundary conditions imposed instead to determine whether a stationary state is convectively unstable. Consider the irregular state of some system. Let λ_g be the largest Lyapunov exponent for the given set of boundary conditions, and λ_p be the largest Lyapunov exponent for periodic boundary conditions imposed instead. If $\lambda_g < 0$ and $\lambda_p > 0$, we can say that the state is convectively chaotic and that λ_p is a measure of that chaos.[2] If $\lambda_g > 0$, the state will be absolutely chaotic. If $\lambda_p \leqslant 0$, the state will not be chaotic.

5. SLUGS

Consider the stationary state of some system and a finite, spatially localized perturbation about this state. Assume that the stationary state of this system is subcritical, meaning that only a perturbation of sufficient amplitude will grow, whereas a perturbation of smaller amplitude will damp. Therefore, if the perturbation is sufficiently large, it will grow and eventually saturate as a result of nonlinearities, producing a localized structure which will be surrounded by the stable stationary state. Note that part of the structure may also be in contact with boundaries. This structure may be regular or irregular, may or may not be traveling, and may or may not be spreading. This localized structure is called a *slug*[9,10,19–21] and therefore I define a slug as a localized structure which is surrounded by a stable stationary state. By stable I mean stable to sufficiently small perturbations.

To make this concept more concrete, Fig. 7 shows a turbulent slug in plane Poiseuille flow (fluid flowing between two parallel plates), looking down on (i.e., perpendicular to) the plates. This slug is surrounded by stable laminar fluid and slowly spreads as it travels downstream. It was initially produced by giving the laminar state a sufficiently large pertur-

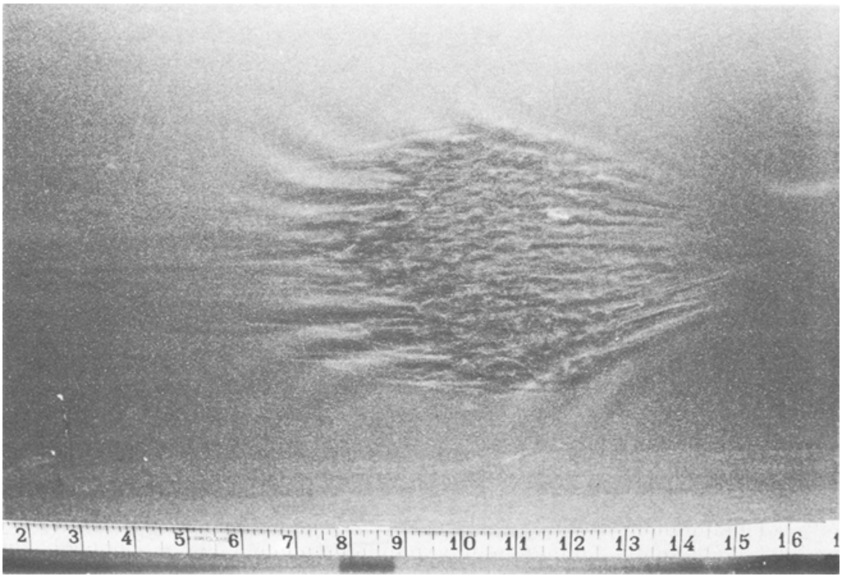

Fig. 7. Turbulent slug in plane Poiseuille flow (fluid flowing between two parallel planes). The slug is surrounded by stable laminar fluid and slowly spreads as it is convected downstream. Reprinted from ref. 20.

bation. If one looks from the side (i.e., parallel to the plates), the slug would be seen to be confined in the perpendicular direction by the plates.

Slugs also occur in the following generalization to Eq. (1)[9]:

$$\frac{\partial \psi}{\partial t} = a\psi - v_g \frac{\partial \psi}{\partial x} + b \frac{\partial^2 \psi}{\partial x^2} - c\,|\psi|^2\,\psi - d\,|\psi|^4\,\psi \tag{10}$$

The only difference between this equation and Eq. (1) is the addition of the quintic term. This term is necessary in order to ensure saturation when $c_r < 0$. For slugs to exist we need $a_r < 0$ so that sufficiently small perturbations will damp, $c_r < 0$ so that sufficiently large perturbations will grow, and $d_r > 0$ so that the perturbation will saturate and form a slug.

Figure 8 shows a slug in Eq. (10) at successive times separated by $t = 20$. The boundary conditions are periodic and $v_g = 0$. The initial state is that shown in Fig. 8a. A small random initial perturbation is superimposed on this initial state in order to break any symmetries—without this random perturbation the slug would stay symmetric about $x = L/2$, even though the slug may be changing chaotically with time. On either side of the slug the solution is stable (i.e., stable to sufficiently small perturbations) since $a_r < 0$.

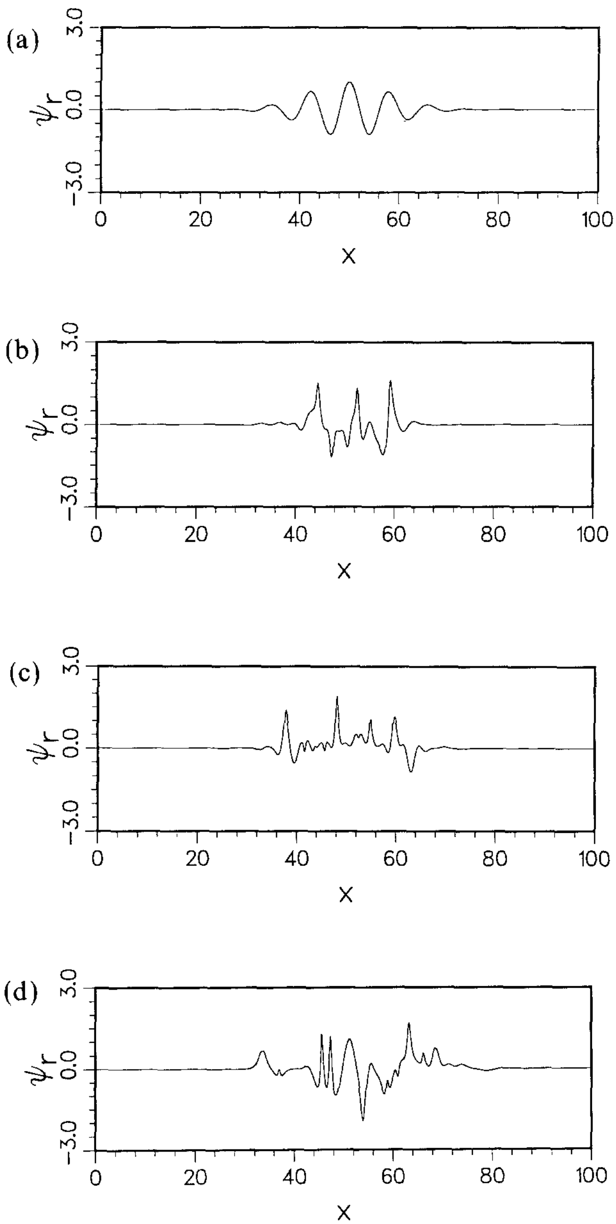

Fig. 8. Turbulent slug in the generalized Ginzburg–Landau equation (10) at times separated by $t = 20$. The parameter values are $a = -0.1$, $v_g = 0$, $b_r = 0.4$, $b_i = -1$, $c_r = -2$, $c_i = 1$, $d_r = 0.5$, and $d_i = 1$. The initial condition was that seen in Fig. 8a. The slug slowly spreads and changes in a chaotic fashion with time.

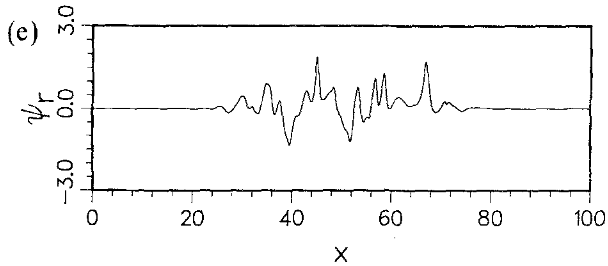

Fig. 8 (*continued*)

We see that the slug spreads slowly and changes in a chaotic fashion with time.

In order to see that the slug is indeed chaotic, Fig. 9 shows a plot of the logarithm of the separation between two nearby states as a function of time. The separation is given by Eq. (7), where $\delta\psi$ was calculated by solving the linear equation

$$\frac{\partial \delta\psi}{\partial t} = a\delta\psi - v_g \frac{\partial \delta\psi}{\partial x} + b \frac{\partial^2 \delta\psi}{\partial x^2} - 2c\,|\psi|^2\,\delta\psi - c\psi^2\delta\psi^* - 3d\,|\psi|^4\,\delta\psi - 2d\psi^3\psi^*\delta\psi^* \tag{11}$$

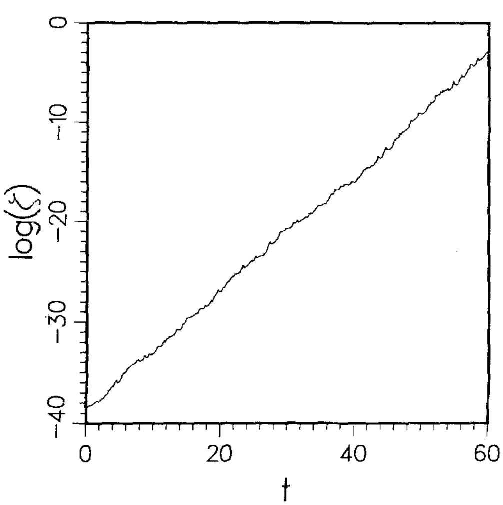

Fig. 9. Separation between two nearby states given by Eq. (7) for the slug seen in Fig. 8, where $\delta\psi$ is given by Eq. (11). The slug seen in Figs. 8b–8e correspond to $t = 0$, 20, 40, and 60, respectively. The states separate exponentially, showing that the slug is chaotic.

along with Eq. (10). Note that the absolute vertical scale is irrelevant since one is solving a linear equation (i.e., one may add any constant to the vertical scale). Also note that plotting the logarithm of the separation[22] is an efficient way to calculate the largest Lyapunov exponent, since typically one does not need as long of a time series as in directly using the definition, and also renormalization is not necessary if one is solving for the perturbation from the linearized equation [e.g., Eq. (11)], since the amplitude of the perturbation can change many orders of magnitude before the computer overflows (approximately 1000 orders of magnitude for a Cray computer). Another advantage is that the degree of chaos may be slowly changing for some systems (corresponding to a change in the average slope of the curve) which would not be observed if one were to apply directly the definition.

For example, for incompressible fluid flow, the Navier Stokes equations are[3]

$$\dot{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\nabla^2\mathbf{u} = -\nabla p + \mathbf{f} \tag{12}$$

and the continuity equation is

$$\nabla \cdot \mathbf{u} = 0 \tag{13}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity field of the fluid, $p(\mathbf{x}, t)$ is the kinematic pressure (i.e., the pressure divided by the density), $\mathbf{f}(\mathbf{x}, t)$ is an external force field (e.g., gravity), and ν is the kinematic viscosity. Let $\delta\mathbf{u}$ and δp be infinitesimal perturbations about $\mathbf{u}$ and p, respectively. These perturbations will satisfy

$$\dot{\mathbf{u}} + (\delta\mathbf{u} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\,\delta\mathbf{u} - \nu\nabla^2\,\delta\mathbf{u} = -\nabla\delta p \tag{14}$$

and

$$\nabla \cdot \delta\mathbf{u} = 0 \tag{15}$$

The separation between two nearby states is

$$\zeta(t) = \left(\int_\Omega \delta\mathbf{u} \cdot \delta\mathbf{u}\, d\mathbf{x}\right)^{1/2} \tag{16}$$

where Ω corresponds to the region over which the separation is being measured, which in many cases corresponds to the entire system. If the system is inhomogeneous with different regions of the flow exhibiting different degrees of chaos, it may be desirable to let Ω be a subset of the system over which the degree of chaos is being measured. Also, for systems with a mean flow velocity it is possible—and, as previously noted, necessary for systems without periodic boundary conditions—to let the region Ω move with the flow. In this way, the degree of chaos for some moving localized structure such as a slug could be measured. The separation ζ from Eq. (16) can be calculated and plotted as a function of time by solving the linear equations (14) and (15) along with Eqs. (12) and (13). The average slope will then give the Lyapunov exponent.

Since a slug is a structure which forms subcritically, a question which arises is how slugs form naturally in open-flow systems such as pipe flow and channel flow. One explanation is that, even though the flow may be subcritical sufficiently far downstream (in fact, fully developed laminar cylindrical pipe flow is linearly absolutely stable for all Reynolds numbers), the flow may be convectively unstable in the inlet region where the flow is not yet fully developed.[9,23,24] For example, the velocity profile in cylindrical pipe flow and plane channel flow changes gradually from essentially flat to parabolic as the distance from the entrance to the pipe or channel increases. Even though the fully developed flow is stable, a portion of the developing flow may be unstable.

Therefore, the following scenario for turbulent slug production and spatiotemporal intermittency in pipe and channel flow was suggested in ref. 9, where Eq. (10) was studied with a varying value for the coefficient a (a_r being positive in the left region to simulate a convectively unstable inlet region, and a_r and c_r being negative in the right region to simulate a subcritical region). External fluctuations are selectively and spatially amplified in the inlet region, giving rise to spatially growing waves. Since the source of the spatially growing waves is random noise, there will be random variations in the amplitude of the spatially growing waves. Further downstream the system is subcritical with two basins of attraction—a laminar basin and a turbulent basin. If a portion of the wave becomes sufficiently large in the inlet region, it will grow into a slug in the subcritical region further downstream (i.e., settle into the turbulent basin); if it does not become sufficiently large in the inlet region, it will damp and form a laminar region (i.e., settle into the laminar basin). The result will be alternating regions of laminar and turbulent flow. Since the variation in amplitude of the spatially growing waves is random, the slugs will form at random intervals in time—resulting in random spatiotemporal intermittency.

It is also possible to have a slug that changes periodically with time and does not spread. Figures 10b and 10c show such a slug at two different times. The initial condition was that seen in Fig. 10a (including a small initial random perturbation). The random initial condition is irrelevant in this case, since it damps instead of grows as in the chaotic case. This slug

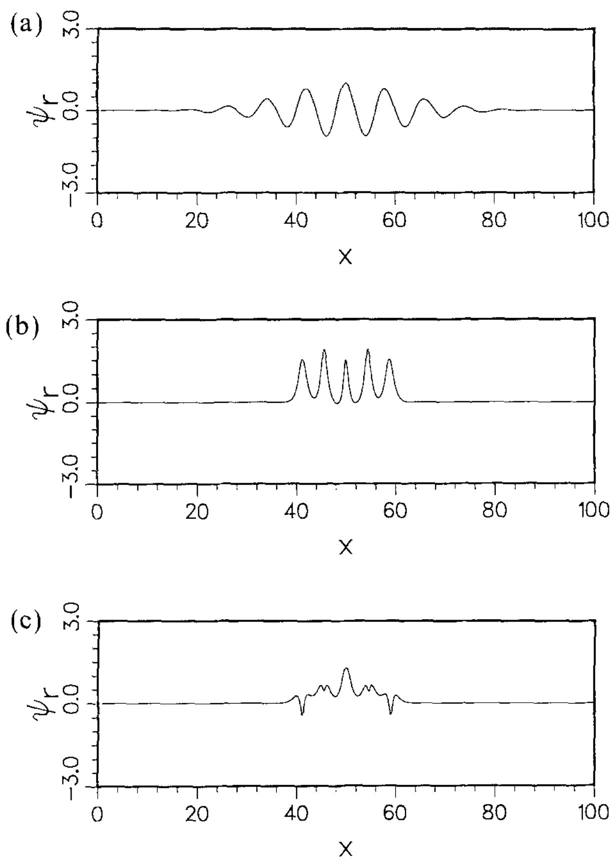

Fig. 10. A slug in Eq. (10) which does not spread and changes periodically with time. The parameter values are the same as those in Fig. 8, except for $b_r = 1$.

oscillates periodically and will remain localized indefinitely with time. This is similar to the slugs which change periodically and do not spread with time that can occur in binary fluid mixtures (see Fig. 11).[10,25]

The fact that the chaotic slug spread with time, whereas the periodic slug does not spread with time, may be understood in the following fashion. The boundary between the slug and the laminar region is not sharp but gradual, with the wave damping exponentially with distance from the edge of the slug. In the chaotic case the amplitude of the wave adjacent to the slug changes in a random fashion with time. Occasionally the amplitude of the wave adjacent to the slug will become sufficiently large so that it will grow (i.e., "pop" into the chaotic basin of attraction), producing an additional contribution to the slug. This random "popping"

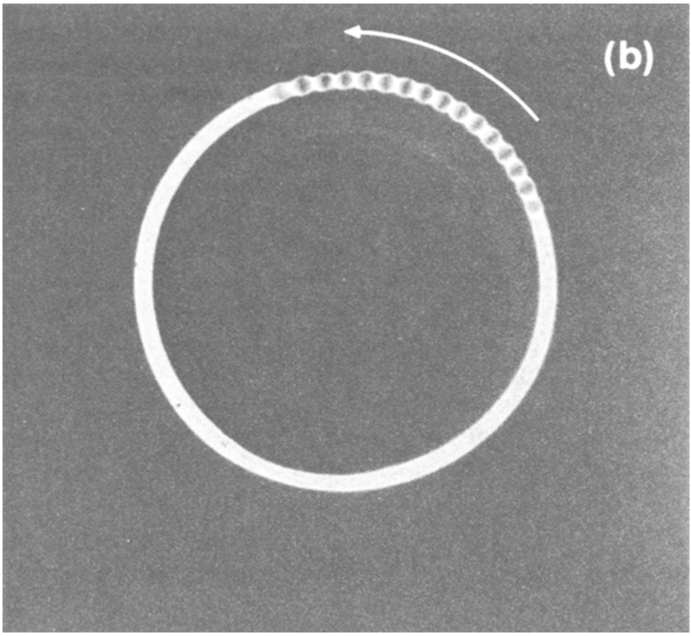

Fig. 11. A periodic nonspreading slug in binary fluid convection. Reprinted from ref. 25.

will cause the slug to spread randomly with time. In the periodic case the wave adjacent to the slug oscillates periodically with time and therefore never changes amplitude and is never large enough to grow. Therefore, in this case the slug will not spread. The spreading mechanism for the chaotic slug discussed above is similar to the turbulent from propagation mechanism in a string of oscillators in which turbulent oscillators contaminate nearby quiescent oscillators.[26]

6. CONVECTIVE SECONDARY INSTABILITIES

As noted in Section 3, the regular portion of the structure broke up as a result of a convective secondary instability. The convective nature of this secondary instability can be seen most clearly by perturbing the left boundary sinusoidally instead of with noise. Figure 12b shows the result. We see that the structure is sinusoidal. Since it is convectively unstable, we know that external noise will play an important role in the dynamics. Figure 12a shows the resulting structure for large times with a small amount of noise introduced at the left boundary in addition to the sinusoidal perturbation. The noise is amplified as it is convected to the right, causing the structure

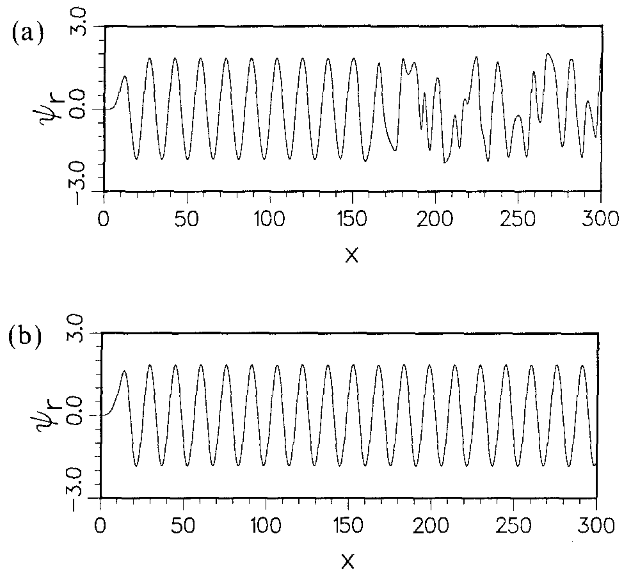

Fig. 12. Plot of ψ_r as a function of x. The left boundary is perturbed sinusoidally, $\psi(0, t) = Ae^{-i\omega t}$, with a frequency of $\omega = 1.111$ and an amplitude of $A = 0.01$. The parameter values are the same as those in Fig. 2. (a) A small amount of noise (noise level $r = 10^{-3}$) is added at the left boundary in addition to the sinusoidal perturbation. (b) No noise is added. With noise the structure breaks up as a result of a secondary convective instability. Without noise the structure is periodic.

to break up at some spatial point. Without noise the structure is periodic; with noise the structure is irregular for sufficiently large x (and in fact can be shown to be convectively chaotic).

It is also possible to have an absolute primary instability and a convective secondary instability. Figure 13c shows a plot of ψ_r as a function of x in the absence of noise under conditions when the state $\psi = 0$ is absolutely unstable. We see that the structure is self-sustained since the state $\psi = 0$ is absolutely unstable. If we now add noise at the left boundary (see Figs. 13a and 13b), we see that the structure breaks up as a result of a convective secondary instability (the larger the noise level, the closer to the left boundary the point of breakup). This shows that external noise can also play an important role in systems for which the stationary state is absolutely unstable, but for which a secondary state is convectively unstable.

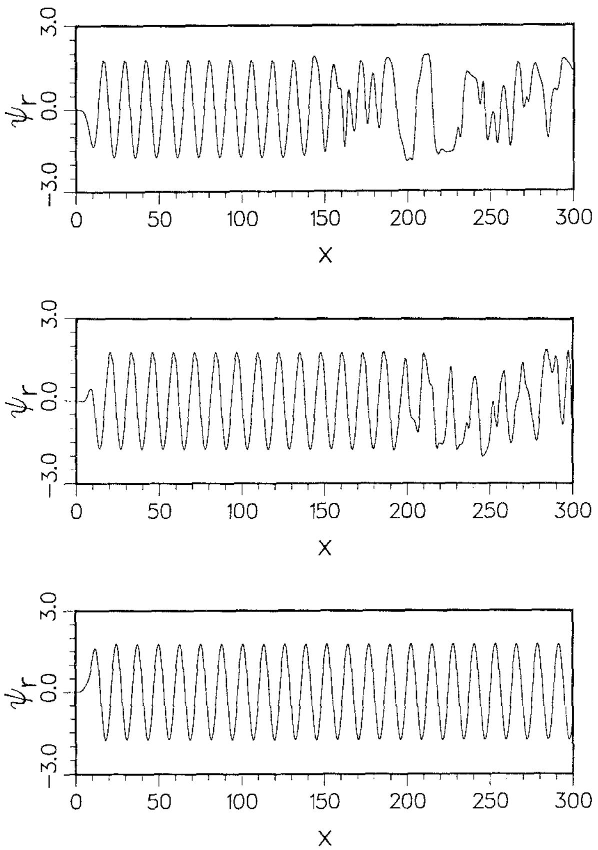

Fig. 13. Plot of ψ_r as a function of x under conditions when the state $\psi = 0$ is absolutely unstable. The parameter values are the same as those of Fig. 2, except that $v_g = 4$. (a) Noise level at the left boundary is $r = 10^{-4}$. (b) Noise level at the left boundary is $r = 10^{-6}$. (c) Noise level is $r = 0$. With noise the structure breaks up as a result of a secondary convective instability. Without noise the structure is periodic. In contrast to Fig. 12, for which the state $\psi = 0$ is convectively unstable, the structure in Fig. 13 is self-sustained.

7. SPATIALLY VARYING INSTABILITIES AND PERIODIC BOUNDARY CONDITIONS

It is also possible for low-level external noise to play an important role in systems with periodic boundary conditions if the system is absolutely stable in one region and convectively unstable in another region.[1] Figure 14 shows results for Eq. (1) with and without noise, where a is positive in the left portion of the system and negative in the right portion of the system. Without noise, a signal which is amplified in the unstable

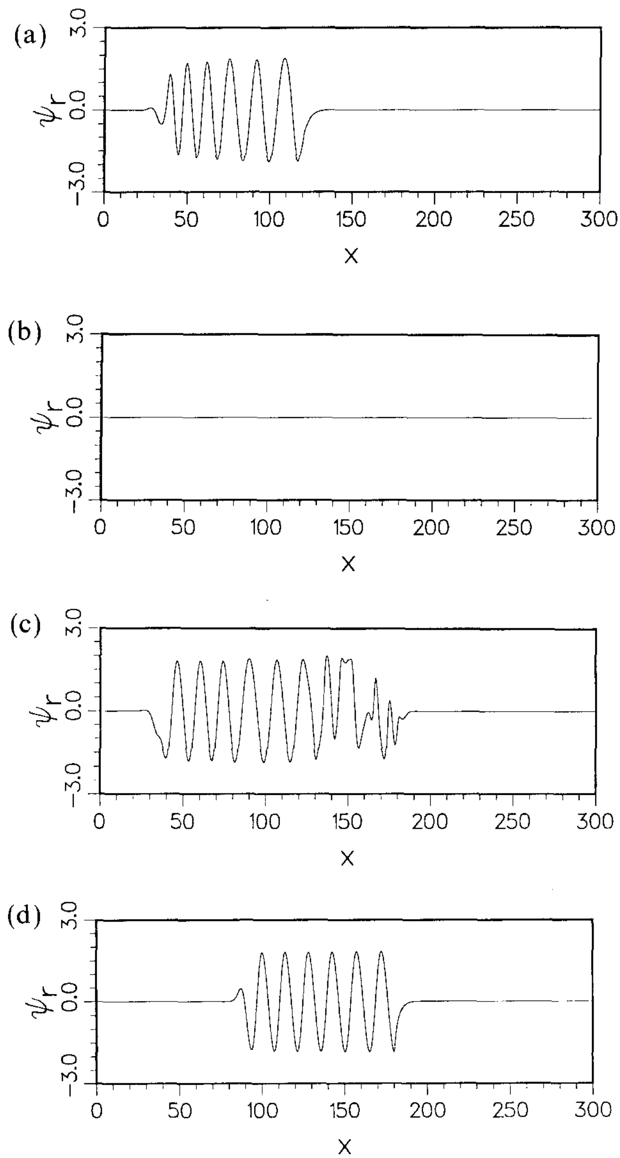

Fig. 14. Plot of ψ_r as a function of x under periodic boundary conditions. The state $\psi = 0$ is convectively unstable for $0 < x < x_1$ and absolutely stable for $x_1 < x < 300$. The parameter values are the same as those of Fig. 2, except that $a = -2$ for $x_1 < x < 300$. Noise of level $r = 10^{-6}$ is introduced at the left boundary in panels a, c, and e. The noise is removed in panels b, d, and f. (a, b) $x_1 = 120$; (c, d) $x_1 = 180$; (e, f) $x_1 = 240$. This demonstrates that low-level external noise can also play an important role in the dynamics of systems with periodic boundary conditions.

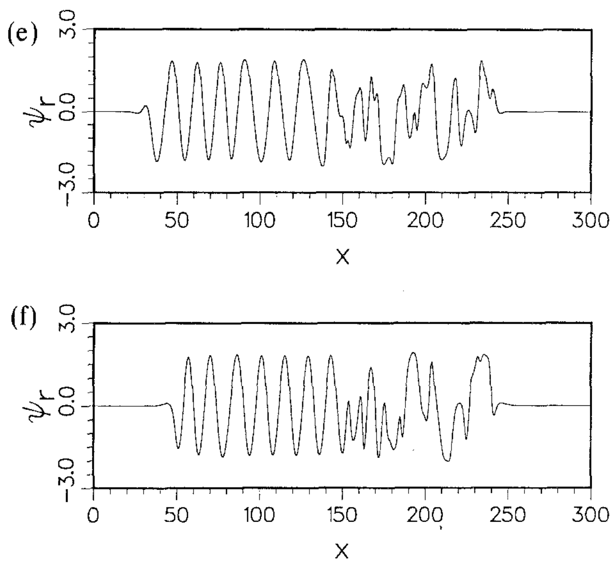

Fig. 14 (*continued*)

region is damped in the stable region and then fed back to the unstable region—since the boundary conditions are periodic—where it is again amplified (see Figs. 14b, 14d, and 14f). In Fig. 14b the signal does not become large enough to be seen before it is damped. In Fig. 14d the signal at the left boundary will be periodic, since the structure is periodic. In Fig. 14f the signal at the left boundary is random in nature since the right of the structure is chaotic. In this case there is random spatiotemporal intermittency similar to the noise-driven case in Section 3. If external noise is added which is larger than the signal at the left boundary, the noise will then play an important role in the dynamics, the behavior being similar to the noise-driven case in Section 3 (compare Figs. 14a, 14c, and 14e with Figs. 14b, 14d, and 14f).

8. COUPLED GINZBURG–LANDAU EQUATION

In some systems, such as binary fluid convection—which consists of two miscible fluids, such as water and alcohol, with a vertical temperature gradient—the system can be reduced to the two coupled Ginzburg–Landau equations[15]

$$\frac{\partial \psi_1}{\partial t} = a\psi_1 - v_g \frac{\partial \psi_1}{\partial x} + b \frac{\partial^2 \psi_1}{\partial x^2} - c\,|\psi_1|^2\,\psi_1 - d\,|\psi_2|^2\,\psi_1 \tag{17}$$

and

$$\frac{\partial \psi_2}{\partial t} = a\psi_2 + v_g \frac{\partial \psi_2}{\partial x} + b \frac{\partial^2 \psi_2}{\partial x^2} - c\,|\psi_2|^2\,\psi_2 - d\,|\psi_1|^2\,\psi_2 \tag{18}$$

These equations were studied in ref. 10. Because the group velocities are opposite in sign, these equations exhibit counterpropagating nonlinear waves similar to those observed in binary fluid convection experiments.[27-29] Because of the nonlinear cross-coupling, the counterpropagating waves will interact and a large number of qualitatively new phenomena occur. For details the reader is referred to ref. 10. However, here I just note that, whereas the linear growth rate coefficient for a single Ginzburg–Landau equation [Eq. (1)] is a_r, the *effective* linear growth rate coefficients for Eqs. (17) and (18) are $\gamma_1 = a_r - d_r\,|\psi_2|^2$ and $\gamma_2 = a_r - d_r\,|\psi_1|^2$, respectively. Therefore, if d_r is positive, the cross-coupling will have a stabilizing effect; if d_r is negative, the cross-coupling will have a destabilizing effect. The cross-coupling can therefore cause a suppression or enhancement of the instability and can also cause a transition between different stability regimes. For example, assume that $d_r < 0$ and that initially $\psi_1 = 0$ and $\psi_2 = 0$ and that this state is convectively unstable [i.e., $0 < a_r < v_g^2 b_r/(4\,|b|^2)$; see Eq. (5)]. If noise is introduced at both boundaries, the noise will be selectively and spatially amplified as it is convected away from the boundaries, giving rise to counterpropagating waves. When the waves reach the opposite boundaries and assuming that d_r is sufficiently negative, the effect of nonzero ψ_2 and ψ_1 will be to increase the effective linear growth rate coefficients γ_1 and γ_2, respectively, so that the system becomes absolutely unstable [i.e., $\gamma_{1,2} > v_g^2 b_r/(4\,|b|^2)$ on the average].

The studies in ref. 10 did not include reflections. As noted in that paper, this can be realized in experiment by having soft boundaries.[30] For example, the distance between the plates in a binary fluid experiment could be gradually changed near the boundaries to cause the waves to damp gradually and therefore eliminate reflections. Therefore, in such a system noise introduced near one of the boundaries should be able to produce a noise-sustained structure similar to that studied in refs. 1 and 2. Also, other types of behavior studied in ref. 10, such as transition between different stability regimes, should be able to be observed.

In addition, quintic terms [as in Eq. (10)] can be added to Eqs. (17) and (18), and the system can exhibit counterpropagating slugs.[10] If the cross-coupling is stabilizing, the slugs can partially annihilate upon interaction. As previously noted, Fig. 10 shows a slug in binary fluid convection. With the proper set of parameter values and initial conditions, it should be possible to produce counterpropagating slugs in binary fluid convection which will partially annihilate upon interaction, assuming a stabilizing

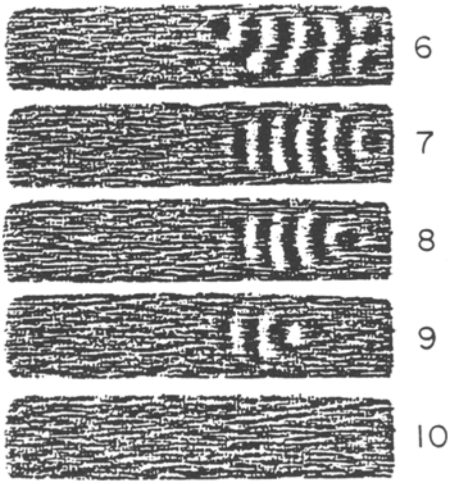

Fig. 15. The "football state" in binary fluid convection. Fluctuations near the left boundary are amplified as they are convected to the right, resulting in the observed structure. Reprinted from ref. 28.

interaction. Also, as noted in ref. 10, the intermittent behavior observed in liquid ^{3}He–^{4}He binary fluid experiments[31] may be the result of the formation of slugs.

Figure 15 shows a state in binary fluid convection called the "football state."[27,28] As pointed out in ref. 10, this state can be understood in terms of convective instability and noise-sustained structure,[1,2] where fluctuations near the left boundary are selectively and spatially amplified as they are convected toward the right. The reason that the left portion of the cell is empty of waves is due to the fact that fluctuations near the left boundary need to be amplified above a certain threshold before they are large enough to be seen.

9. OTHER SYSTEMS AND CONCLUSIONS

As noted in the introduction, any system with nonzero group velocity will be convectively unstable slightly above criticality. Therefore, the behavior reviewed and studied in this paper should be very common in nature. A few physical systems which have already been mentioned are open flow systems such as pipe flow, channel flow, and flat-plate flow. In these systems external noise near the entrance to the pipe or channel, or near the leading edge of the plate, will be selectively and spatially amplified, resulting in spatially growing waves and random spatiotemporal

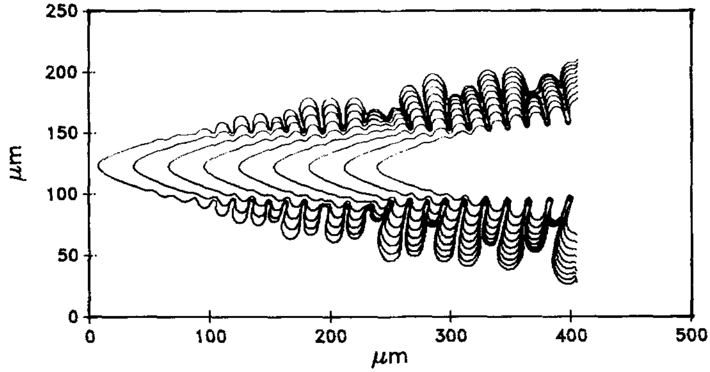

Fig. 16. Side branching in a dendrite of NH_4Br. External noise near the tip of the dendrite is selectively and spatially amplified as it is convected away from the tip along the sides of the dendrite, resulting in the observed pattern.

intermittency. Another system which has been mentioned is binary fluid convection. This system has the potential to produce all the types of behavior observed in a single Ginzburg–Landau equation and two coupled Ginzburg–Landau equations, particularly if soft boundaries are imposed to eliminate reflections.

Another prototype equation—other than the Ginzburg–Landau equation—which has been recently studied is the Kuromato–Sivashinsky equation.[32] This equation is related to such systems as flow down an inclined plane, propagation of flame fronts, and autocatalytic chemical reactions. This equation was shown to be convectively unstable for a certain range of parameter values and to exhibit noise-sustained structure and spatiotemporal intermittency.

Another system (mentioned in Section 3) for which it is now clear that external noise and the concept of convective instability, and therefore the pattern selection mechanism of refs. 1 and 2, are important is side branching in dendrites and fingers.[33–36] Figure 16 shows a dendrite growing from a solution of NH_4Br. Noise near the tip of the dendrite is selectively and spatially amplified as it is convected away from the tip along the sides of the dendrite. This selective and spatial amplification of noise serves as the pattern selection mechanism responsible for the wavelength of the side branches. Also, because of the random nature of the noise, the pattern which forms is not completely regular, but has irregularities.

In the introduction it was mentioned that the behavior reviewed and studied in this paper is expected to be very common in nature. In closing I mention one final system which is far afield from the systems previously mentioned. A simple model which describes traffic flow is[37,38]

$$\frac{d^2x_n(t+T)}{dt^2}=\frac{\lambda_0}{x_{n-1}(t)-x_n(t)}\left(\frac{dx_{n-1}(t)}{dt}-\frac{dx_n(t)}{dt}\right)$$

where x_n is the position of the nth car and T is the time it takes the $(n+1)$th car to react to changes in velocity of the nth car. This equation can be shown, for the proper parameter values, to be convectively unstable. Therefore, fluctuations in the speed of the lead car will be selectively and spatially amplified as they are convected down the string of cars, giving rise to waves in the speeds of the following cars. It would be interesting to study this and similar equations in the light of the work reviewed and studied here.

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